

Entropy of String States at fixed Mass and Size

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Abstract

We provide formulas for the entropy of free-string states depending on their mass, charges and size, both in bosonic and superstring theory (IIA or IIB). We properly define these quantities in full-fledged string theory. We then investigate the corrections to the entropy due to self-interactions of the string for states with fixed mass, charge and size, both for BPS and non-BPS configurations. Again, the analysis is performed using string theory techniques.

Contents

1	Introduction	1
2	The String-Black Holes correspondence principle	3
3	The microcanonical ensemble	5
4	States with no charge	7
4.1	The setup	7
4.2	The String Spatial Distribution.	9
4.2.1	Corrections	13
4.3	Number of string states of a given mass and size.	14
5	States carrying Neveu-Schwarz charges	16
5.1	BPS states	18
6	One-loop corrections	18
6.1	States with no charge	18
6.2	Loop corrections for BPS states	23
7	Conclusions	25
8	Acknowledgements	25

1 Introduction

Black holes obey a set of laws that are formally identical to the thermodynamical ones:

1. $\kappa_s = \text{constant}$ on horizon ($\kappa_s = \text{surface gravity}$)
2. $\delta M = \frac{\kappa_s}{8\pi G_N} \delta A + \omega \delta J + \phi \delta q$
 ($M = \text{black hole mass}$, $A = \text{horizon area}$, $\omega = \text{angular velocity at horizon}$,
 $J = \text{angular momentum}$, $\phi = \text{electric potential}$, $q = \text{electric charge}$
 $G_N = \text{Newton's constant}$)
3. $\delta A > 0$ ¹

¹This law is violated by Hawking radiation, and therefore substituted by a generalized law stating that the area of the black holes plus the entropy of the universe do not decrease in a physical process.

4. $\kappa_s = 0$ impossible for any physical process.

It is tempting to relate, then, A to an “entropy” S , and κ_s to a temperature T . This is meaningless in classical theory, since black holes do not radiate and it is not possible to associate a temperature to them. At this level is therefore necessary a quantum theory. In fact, Hawking radiation determines

$$\kappa_s = 2\pi T$$

and then

$$S = \frac{A}{4G_N} + \dots$$

where the dots stand for corrections due to higher curvature terms and quantum contributions as well².

In the modern interpretation, we think of classical gravitational black holes as a coarse-grained description of a quantum system³. This last description is the one that should provide us not only with an interpretation of black holes’ laws, but also with the possibility of deriving them from first principles.

We will focus in particular on the entropy. According to Bekenstein principle, the black hole’s one is proportional to the area of the horizon (plus corrections); for a quantum statistical ensemble, instead, the entropy is defined as the logarithm of the number G of microstates:

$$S = \ln(G). \tag{1}$$

Relating the two definitions represents the entropy issue. It is necessary to individuate the correct ensemble of microstates accounting for S : we need therefore a quantum gravity theory. String theory represents probably the best candidate nowadays for such a theory and it has a general principle (known as the “String-Black holes correspondence principle”) individuating those microstates. In the case of supersymmetric BPS configurations at fixed tree-level mass, their counting at the level of free string (string coupling

²The more general classical formula for the entropy of a black hole, including higher derivative terms, is Wald’s one [1].

³This is one of the possible resolution of the problems posed by classical black holes. As an example it is typical of the “fuzzball” proposal, which sees the black hole solution as the coarse-grained description of many microstates’ solutions which have no horizon. Other interpretations of black holes which do not includes processes of averaging and coarse-graining are in fact possible, but in general they have to cope with the problem of explaining how an horizon and trapped surfaces can arise from the pure quantum states accounting for the black hole microstates [2].

$g_c = 0$) does not receive corrections, in the non supersymmetric case, instead, quantum corrections must be taken in consideration.

In this work we want to focus on these effects on the string theory side (microstates). We consider closed string, and perform our analysis both in the bosonic and in the superstring theory (type IIB or IIA). We will deal both with states carrying no charges and with states carrying charges of the Neveu-Schwarz type. We begin in Section 2, by reviewing the String-Black Holes Correspondence Principle. Special emphasis is given to the role played by the value of the horizon radius of the black hole and the corresponding requirements for the size of the string microstates.

It is therefore interesting to find the entropy of string microstates depending on both mass and size (concept that we will define properly in the following). This task is not easily solved because it is not straightforward, in the quantum theory of strings, to define an operator measuring the (average) size of string states. The main topic of this work will therefore be to compute such entropy in a well-defined way. Furthermore, the free-string entropy as a function of mass and size will now receive corrections in both the non-BPS and the BPS cases (renormalization of the radius).

We discuss and specify our statistical ensemble of closed string states in section 3, and in 4 we investigate the spatial distribution and the number of microstates with zero charge at fixed squared mass and “size”. At the end of the section we provide formulas for the entropy of single free (bosonic and super-)string states constrained both in mass and in size.

We extend our results to string states carrying Neveu-Schwarz charges (winding and Kaluza-Klein mode numbers) in section 5, and study BPS states as well.

In section 6 we study the one-loop corrections: we propose a method for implementing the constraint on the size of string states and investigate how this affects the one-loop amplitude. The results are obtained by evaluating full-fledged (super)string path integrals. There are important differences between the non-BPS and the BPS case: we elucidate them and treat both cases, separately. Finally, we comment and conclude.

2 The String-Black Holes correspondence principle

String theory and black holes’ physics set two characteristic length scales:

$$\begin{array}{ll} R_{bh} & \text{the black hole horizon radius (Schwarzschild radius)} \\ l_s = \sqrt{\alpha'} & \text{the string length scale} \end{array}$$

so that

$$\left\{ \begin{array}{ll} \text{if } R_{bh} \gg l_s & \text{general relativity description is reliable} \\ \text{if } R_{bh} \lesssim l_s & \text{strings feel space-time as flat, } \alpha'\text{-corrections are important,} \\ & \text{string theory description is reliable} \end{array} \right.$$

The ‘‘String-Black Holes Correspondence Principle’’ states that a black hole is described by an ensemble of excited string and/or D-brane states (depending on the type of charges the black hole possesses) when $R_{bh} \sim l_s$.

There are two possible interpretations of the Principle (for simplicity we consider now the case without charges):

- a **physical process** (Hawking radiation) where the black hole decreases its mass, therefore reducing the value of its Schwarzschild radius $R_{bh} \sim (G_N M)^{\frac{1}{d-2}}$ until $R_{bh} \sim l_s$ where a transition to an excited string states takes place;

$$\text{in this case } \begin{cases} g_c & \text{is fixed} \\ M & \text{varies} \end{cases}$$

- **two complementary descriptions** valid in different regimes at equal mass;

$$\text{in this case } \begin{cases} g_c & \text{varies} \\ M & \text{is fixed} \end{cases}$$

The possibility of equating the black hole entropy (proportional to a power of its mass) and the string one (proportional to the square root of its mass) relies on the fact that the first is constant in Plank units, the second in string ones and therefore the entropies match at a determined value of the string coupling. At the transition point it is found that, in units of α' [4],

$$R_{bh} \sim l_s \Rightarrow g_c \sim M^{-\frac{1}{2}} \quad (2)$$

independently of the number of dimensions. Since we are to consider very massive string states, this value for the string coupling turns out to be sufficiently small to allow perturbation theory.

We would expect that only states whose size is of the same order of the black hole horizon radius can be related to the black hole at the transition

³We are in $d = D - 1$ spatial dimensions, and we relate Newton’s constant to the string length as $G_N \sim g_c^2 (\alpha')^{d-1}$ at small closed string coupling g_c , see [3].

point⁴. It is therefore interesting to find the entropy of string microstates depending on both mass and “size”.

In this work we will determine the number (and therefore the entropy) of perturbative (super)string states depending on their mass and size at tree-level and we will investigate the corrections to their entropy due to the self-interactions of the string. We will then consider states carrying charges, and extend the results to that case.

In the past, a few attempts have been made to study such issues: [3], [5] (see also [6]). In [5], it was employed a thermal scalar formalism, interpreting the size of the bound states of a certain scalar field as the size of the excited string. The thermal scalar is a formal device capable to give us some statistical information about the string system (string gas). Nevertheless its relation to the string states remains open. In particular, an Hamiltonian interpretation for its degrees of freedom seems to question an identification between its states and the string spectrum (see [7]).

The approach followed in [3] was more directly linked to a model of (bosonic open) strings, but the computations were performed within a simplified toy model, believed to be valid in a large number of dimensions ($d \gg 1$), not taking into account Virasoro constraints.

We will perform our calculations in full-fledged string theory.

The computation of the entropy of strings is ultimately connected also with the Hagedorn transition in string theory, but we will deal with single-string entropy and therefore our results do not apply directly.

Our conventions, here and in the following are:

$$\begin{aligned} \alpha' &= 4, & D &= d + 1 \text{ large space-time dimensions} \\ M_{\text{tree}} &= \sqrt{N} & &= \text{bare mass of the string state} \\ g_c/g_o &= & &= \text{closed/open string coupling.} \end{aligned}$$

Furthermore, objects with “*c*” subscript will refer to closed strings, whereas those with an “*o*” subscript will relate to open strings.

3 The microcanonical ensemble

We want to determine statistical properties of massive string states, in particular concerning their spatial distribution. We will use the *microcanonical*

⁴Consider for example the gravitational binding to mass ratio, or, for the case of the “fuzzball” proposal, the fact that the metric sourced by the microstates must differ from the one of a black hole only at distance lower than the horizon radius.

ensemble. Ensembles are defined by density matrices: the microcanonical one has the form⁵

$$\rho_E = a_E \delta(E - \hat{H}) \quad (3)$$

where \hat{H} ⁶ is the Hamiltonian of the system and a_E ensures the normalization of the density matrix:

$$\text{tr}[\rho_E] = 1 \quad (4)$$

when traced over the states.

We can try to modify the traditional microcanonical ensemble, fixing the value of other observables, in order to investigate different statistical properties of the system. Considering a discrete observable with associated operator \hat{Q} , we can define the density matrix:

$$\rho_{E,Q} = a_{E,Q} \delta(E - \hat{H}) \delta(Q - \hat{Q}). \quad (5)$$

The quantity

$$\begin{aligned} G(E, Q) &= \text{tr}[\delta(E - \hat{H}) \delta(Q - \hat{Q})] \\ &= \sum_{\phi} \langle \phi | \delta(E - \hat{H}) \delta(Q - \hat{Q}) | \phi \rangle \end{aligned} \quad (6)$$

gives the number of states having the values E , Q for the chosen observables. It is, therefore:

$$a_{E,Q} = G(E, Q)^{-1} \quad (7)$$

If Q represents a continuous observable, we need further to specify a small interval δQ (uncertainty) around the value of the observable we are interested in, and define:

$$\begin{aligned} \rho_{E,Q,\delta Q} &= a_{E,Q,\delta Q} \delta(E - \hat{H}) (\theta(Q + \delta Q - \hat{Q}) - \theta(Q - \hat{Q})) \\ &= a_{E,Q,\delta Q} \delta(E - \hat{H}) \int_Q^{Q+\delta Q} \delta(Q - \hat{Q}). \end{aligned} \quad (8)$$

Once again, tracing the density matrix over the states of the system, yields the number $G(E, Q, \delta Q) = a_{E,Q,\delta Q}^{-1}$ of microstates having values E for the energy, and $Q < Q_i < Q + \delta Q$ ⁷ for the other observable. We will let $\delta Q \rightarrow 0$, so that we can write

$$\rho_{E,Q} = G_{E,Q} \delta(E - \hat{H}) \delta(Q - \hat{Q}). \quad (9)$$

⁵ The expressions for the density matrix are meaningful when applied to the states of a system; with that understanding our notation with Dirac's delta functions is clear.

⁶From now on a $\hat{}$ will distinguish an operator from its value(s).

⁷Here, i runs over the set of microstates.

Our microcanonical ensemble will be defined by fixing the values

- of the level number operator $\hat{N} \equiv -\hat{p}^2$
- of the operator (to be defined) measuring the *size* of the string.

In this way, we will be able to count the number of states $G(N, R)$ with fixed squared mass N and size R , whose logarithm will yield the entropy we are looking for.

A problem arises: it is difficult to define an operator whose (average) value, when applied to a string state, represents its (average) size, satisfying all the constraints of the theory (superconformal or conformal constraints, or BRST constraints in the various quantization procedures) or being computationally manageable (in light-cone gauge), see [3, 8]. In order to cope with this problem, we follow a somehow roundabout procedure.

In sections 4, 5 we do this for free string states, whereas in section 6, we address self-interacting strings.

4 States with no charge

4.1 The setup

As an illustration of the difficulties in defining an operator to measure the *size* of a string state⁸, let us discuss the most natural choice, which will also clarify what we mean with the term *size*. Consider taking the quantized version of the classical average (squared) size of a string:

$$\hat{R}^2 = \frac{1}{\Delta\sigma_+ \Delta\sigma_-} \int_0^{\Delta\sigma_+} \int_0^{\Delta\sigma_-} (X^O(\sigma_+, \sigma_-))^2 \quad \sigma_{\pm} = \sigma \pm \tau, \quad (10)$$

where X^O represents the projection of the oscillator part of the string coordinate orthogonally to the center of mass momentum of the string. Three evident issues regarding such operator are:

- its definition is gauge-dependent,
- the operator has a zero-order contribution proportional to

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad (11)$$

which needs to be interpreted and regularized (see [8]),

⁸We consider both pure and mixed states.

- the insertion of this operator in a path-integral is problematic because it does not commute with the BRST operator, or, when using Light-Cone gauge, poses ordering problems⁹.

The approach that we will choose solves all these issues. We will adopt a physical way of measuring the spatial distribution of an object: *the spatial distribution of an object is obtained from the scattering of other (light) probes off it in an elastic limit*. This will lead us to correctly define the density matrix for our ensemble.

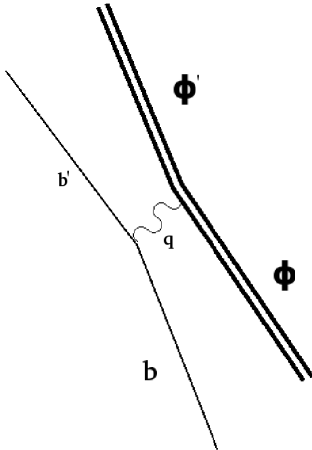


Figure 1: *Scattering process $b + \phi \rightarrow b' + \phi'$. At low momentum transfer the process is dominated by the massless channel, which is represented here.*

Consider, indeed, the Born approximation for the amplitude of a process

$$A = b + \phi \rightarrow b' + \phi', \quad q \equiv \text{exchanged momentum} \quad (12)$$

(see figure 1). At low momentum transfer, in the elastic limit, this is given by

$$A \sim V(q^2) \dots \sim \frac{F_b(q^2)F_\phi(q^2)}{q^2} \quad (13)$$

and $F_i(q^2)$, m_i can be interpreted respectively as the *form factor* and the mass of the particle i . In particular, for elastic scattering (when $q^2 = \vec{q}^2$), the form factor represents the Fourier transform of the spatial distribution $\mu_i(x)$:

$$F_i(\vec{q}^2) = \int d^d x e^{-i\vec{q} \cdot \vec{x}} \mu_i(\vec{x}) \quad (14)$$

⁹Consider the part of the operator proportional to $X^+ X^-$, where X^- is a quadratic function of the transverse coordinates.

According to the nature of the probe b , we obtain different distributions: mass distribution, charge distribution In the following we will consider the *mass distribution*, choosing b to be a graviton (actually a superposition of graviton, dilaton and Kalb-Ramond field). Our targets ϕ will be closed string states extended in the large uncompactified dimensions, though the analysis can be carried over for open strings as well¹⁰.

4.2 The String Spatial Distribution.

The string theory formula¹¹ for the amplitude (S-matrix) for the process (12) is:

$$A_{\text{closed}} = g_c^2 \int d^2 z \langle \phi | V(k', 1) V(k, z) | \phi \rangle \quad (15)$$

Let us discuss for the moment an ensemble of states with fixed squared mass N only. We then need to trace over the density matrix¹²

$$\rho_N = \frac{1}{G_c(N)} \delta(N - \hat{N}) = \frac{1}{G_c(N)} \oint \frac{dw}{w^{N+1}} w^{\hat{N}} \quad (16)$$

so that:

$$A_{\text{closed}} = g_c^2 \int d^2 z \text{tr}[V(k', 1) V(k, z) \rho_N \tilde{\rho}_N], \quad (17)$$

where objects with a tilde refer to the anti-holomorphic (left-moving) sector.

We consider separately the cases of the bosonic string and the superstring. We will find out that the computations are very similar. We therefore discuss first and at length the superstring, and leave the bosonic one at the end.

The superstring.

As we said, our probe consists of a superposition of the graviton, the dilaton and the Kalb-Ramond field, represented by the vertex operator:

$$V(k, z) = \frac{2}{\alpha'} e^{ik \cdot X} (\xi \cdot \partial X - \frac{i}{2} \xi \cdot \psi k \cdot \psi) (\tilde{\xi} \cdot \bar{\partial} X - \frac{i}{2} \tilde{\xi} \cdot \tilde{\psi} k \cdot \tilde{\psi}) \quad (18)$$

¹⁰See [9] for a study of the size of open string states on the leading Regge trajectory using tachyons and photons as probes.

¹¹Note that this formula, and therefore the result that we will obtain, as we will discuss in the following are very different from the formulas in [10]. In our case the interpretation of $F_N(q^2)$ as a form factor is justified, according to scattering theory, whereas in [10] it could not be accepted, and indeed a different interpretation of the results was proposed in [11]

¹² $G_c(N)$ is the number of states at mass level N , the trace is over *physical* string states.

where we have (formally) written the polarization tensor as $\xi_{\mu\nu} = \xi_\mu \tilde{\xi}_\nu$ and X, ψ are, respectively, the space-time string bosonic and fermionic coordinates. We perform our computations in the gauge $\xi_{00} = 0$.

We will make use of the relation (Kawai, Lewellen and Tye, [12])¹³:

$$A_c(1234; \alpha', g_c) = \frac{\pi i g_c^2 \alpha'}{g_o^4} \sin(\pi \alpha' t) A_o(s, t; \frac{\alpha'}{4}, g_o) \tilde{A}_o(t, u; \frac{\alpha'}{4}, g_o). \quad (19)$$

where s, t, u are Mandelstam variables.

The amplitude that we will compute is therefore:

$$A_o(s, t; 1, g_o) = g_o^2 \int dy \text{tr}[V_{\text{open}}(k', 1) V_{\text{open}}(k, y) \rho_N] \quad (20)$$

with

$$V_{\text{open}}(k, y) = \frac{1}{\sqrt{2\alpha'}} e^{ik \cdot X(y)} (iy \xi \cdot \partial_y X(y) + 2\alpha' k \cdot \psi(y) \xi \cdot \psi(y)). \quad (21)$$

We will consider the limit $t \equiv -q^2 = -(k + k')^2 \rightarrow 0$. The lowest terms of the amplitude in this limit can be calculated using the OPE (see [10]):

$$V_{\text{open}}(k', 1) V_{\text{open}}(k, y) \underset{y \rightarrow 1}{\sim} 2\xi \cdot \xi' (1 - q^2) (1 - y)^{2k' \cdot k - 2} y^{2k \cdot \hat{p}} e^{iq \cdot \hat{X}_O(1)} e^{iq \cdot \hat{x}} \quad (22)$$

where \hat{X}_O indicates the oscillator part of X . The amplitude factorizes as:

$$A_{\text{open}} = g_o^2 A_o^{\text{zero modes}} A_o^{\text{oscillators}} \quad (23)$$

By writing $y = e^{-\epsilon}$ with $\epsilon \rightarrow 0$, we find the result¹⁴:

$$A_o^{\text{zero modes}} = - \int d\epsilon \epsilon^{q^2 - 2} e^{-\epsilon(2k \cdot p + 1)} (1 - q^2) \quad (24)$$

$$\underset{q^2 \rightarrow 0}{\sim} \frac{(2\sqrt{N}E)^2}{q^2} \sqrt{F_b(q^2, E)} (2\sqrt{N})^{-q^2}. \quad (25)$$

where we have defined $F_b(q^2, E) \equiv e^{-2q^2 \ln(E)}$ and $E \equiv k^0$.

Therefore:

$$A_c(1234; 4, g_c) \sim \pi^2 i g_c^2 \frac{(2\sqrt{N}E)^4}{q^2} F_b(q^2, E) (2\sqrt{N})^{-2q^2} A_o^{\text{oscillators}} \tilde{A}_o^{\text{oscillators}} \quad (26)$$

¹³Here we have explicitly written the α' squared length in order to present clearly the formula. Remember that eventually in the computations we will always set $\alpha' = 4$.

¹⁴We need to perform the same analytical continuation as for the Veneziano amplitude, as usual in these representation of the string amplitudes.

with

$$A_o^{\text{oscillators}} = \text{tr}[e^{iq \cdot X_{O(1)}} \rho_N] \quad (27)$$

and we have expanded $\sin(-\pi t) \sim -\pi t \sim \pi q^2$.

According to the results and the discussion in [10, 13], we identify $F_b(q^2, E)$ with the form factor for the probe b .

It is now straightforward to read the form factor for the target ϕ at squared mass N :

$$\begin{aligned} F_N(q^2) &= N^{-q^2} A_o^{\text{oscillators}} \tilde{A}_o^{\text{oscillators}} \\ &= \frac{N^{-q^2}}{G_c(N)} \oint \frac{dw}{w^{N+1}} \oint \frac{d\tilde{w}}{\tilde{w}^{N+1}} \frac{g(w)g(\tilde{w})}{(f(w)f(\tilde{w}))^{d-1}} e^{-2q^2 \sum_{n=1}^{\infty} \frac{w^n}{n(1-w^n)} + \frac{\tilde{w}^n}{n(1-\tilde{w}^n)}} \end{aligned} \quad (28)$$

where

$$g(w) = \left(\frac{1}{\sqrt{w}} g_3(w)^{d-1} - \frac{1}{\sqrt{z}} g_4(w)^{d-1} + g_2(w)^{d-1} \right) \quad (29)$$

$$f(w) = \prod_{n=1}^{\infty} (1 - w^n) \quad g_3(w) = \prod_{r=\frac{1}{2}}^{\infty} (1 + w^r) \quad (30)$$

$$g_4(w) = \prod_{r=\frac{1}{2}}^{\infty} (1 - w^r) \quad g_2(w) = \prod_{r=0}^{\infty} (1 + w^r) . \quad (31)$$

We compute the loop-integrals by saddle point approximation for large N , finding

$$\ln(w) \sim -\frac{\pi}{\sqrt{N}} \sqrt{\frac{d-1}{4} - \frac{q^2}{3}} \quad (32)$$

and similarly for \tilde{w} .

Therefore, considering the elastic limit:

$$\begin{aligned} F_N(\vec{q}^2) &\underset{N \rightarrow \infty}{\sim} \frac{e^{4\pi\sqrt{N}\sqrt{\frac{d-1}{4} - \frac{\vec{q}^2}{3}}}}{G_c(N)} \pi^d \left(\frac{(d-1)}{4} - \frac{\vec{q}^2}{3} \right)^{\frac{d}{2}} N^{-\frac{d+2}{2}} \\ &\underset{\vec{q}^2 \rightarrow 0}{\sim} e^{-\frac{4\pi}{3}\sqrt{\frac{N}{d-2}}\vec{q}^2} \\ &\underset{N \rightarrow \infty}{\sim} \end{aligned} \quad (33)$$

where in the last line we have simplified the result with

$$G_c(N) \sim e^{2\pi\sqrt{N(d-1)}} \pi^d \left(\frac{d-1}{4} \right)^{\frac{d}{2}} N^{-\frac{d+2}{2}} . \quad (34)$$

Finally, the average radius is

$$\langle r^2 \rangle = -2d \partial_{\vec{q}^2} F_N(\vec{q}^2) |_{\vec{q}^2=0} = \frac{8\pi d}{3} \sqrt{\frac{N}{d-1}} \quad (35)$$

and the mass distribution

$$\mu_N(\vec{x}) = \frac{1}{(2\pi)^d} \int d^d q e^{i\vec{q} \cdot \vec{x}} F_N(\vec{q}^2) = \left(\frac{3}{16\pi^2} \sqrt{\frac{d-1}{N}} \right)^{\frac{d}{2}} e^{-\frac{3}{16\pi} \sqrt{\frac{d-1}{N}} \vec{x}^2}. \quad (36)$$

The bosonic string.

The case of the bosonic string follows the same steps. A few things are different:

- the vertex operator for the probe now is:

$$V_{\text{open}}(k, y) = \frac{1}{\sqrt{2\alpha'}} e^{ik \cdot X(y)} (iy\xi \cdot \partial_y X(y)). \quad (37)$$

- due to the absence of fermionic excitations, the number of closed string states at fixed large mass squared N is

$$G_c(N) \sim e^{4\pi\sqrt{N}} \sqrt{\frac{d-1}{6}} \pi^d \left(\frac{(d-1)}{6} \right)^{\frac{d}{2}} N^{-\frac{d+2}{2}}; \quad (38)$$

- the integral (24) becomes now:

$$A_o^{\text{zero modes}} = - \int d\epsilon \epsilon^{q^2-2} e^{-\epsilon(2k \cdot p + 1)}. \quad (39)$$

Namely, we see that the integral would be divergent also for $q^2 = 1$, corresponding to the exchange of a tachyon. But we are considering the limit $q^2 \rightarrow 0$, picking out the graviton pole, so that

$$A_o^{\text{zero modes}} \underset{q^2 \rightarrow 0}{\sim} \frac{(2\sqrt{N}E)^2}{q^2} \sqrt{F_b(q^2, E)} (2\sqrt{N})^{-q^2} \quad (40)$$

as for the superstring. Therefore we obtain:

- form factor

$$F_N(\vec{q}^2) \underset{\frac{\vec{q}^2}{N} \rightarrow 0}{\sim} e^{-2\pi \sqrt{\frac{2N}{3(d-1)}} \vec{q}^2}, \quad (41)$$

- the average radius

$$\langle r^2 \rangle = 4\sqrt{2} \pi d \sqrt{\frac{N}{3(d-1)}}, \quad (42)$$

- the mass distribution

$$\mu_N(\vec{x}) = \left(\frac{1}{8\pi^2} \sqrt{\frac{3(d-1)}{2N}} \right)^{\frac{d}{2}} e^{-\frac{1}{8\pi} \sqrt{\frac{3(d-1)}{2N}} \vec{x}^2}, \quad (43)$$

being intended that the number of extended spatial dimensions now can go up to $d = 25$, not only up to 9 as for the superstring.

4.2.1 Corrections

We show here how the lowest terms in the expansion for $y \rightarrow 1$ in (22), dominate the amplitude (20) and the result is safe against possible corrections in a determined kinetic and mass range.

The superstring.

Without any approximations, the amplitude (20) is given by

$$A_o(s, q^2; 1, g_o) \sim \frac{g_o^2}{G_o(N)} \int d\epsilon \oint \frac{dw}{w^{N+1}} \frac{g(w)}{f(w)^{d-1}} e^{-\epsilon(2k \cdot p + 1)} \psi(\epsilon, w)^{q^2} \quad (44)$$

$$\times [-2\partial^2 \epsilon \ln(\psi(\epsilon, w)) + \chi(\epsilon, w)]$$

with

$$\psi(\epsilon, w) = (1 - e^{-\epsilon}) \prod_{n=1}^{\infty} e^{-q^2 \frac{w^n}{n(1-w^n)} (e^{n\epsilon} + e^{-n\epsilon})} \quad (45)$$

$$\partial_\epsilon^2 \ln(\psi(\epsilon, w)) = \sum_{n=1}^{\infty} n e^{-\epsilon n} + \sum_{n=1}^{\infty} \frac{n w^n}{(1 - w^n)} (e^{n\epsilon} + e^{-n\epsilon}) \quad (46)$$

$$\chi(\epsilon, w) = 2q^2 \sum_{s=2}^4 \left(\frac{\theta_s(0)}{\theta_2(0)} \right)^{\frac{d-1}{2}} \frac{\theta_s(\epsilon)^2 \theta_1'(0)^2}{\theta_1(\epsilon)^2 \theta_s(0)^2}, \quad (47)$$

where we have written $y = e^{-\epsilon}$. Note that $\theta_s(z) \equiv \theta_s(\frac{z}{2\pi i}, \frac{\ln(w)}{2\pi i})$ in the usual notation, where the θ_s 's are the Theta functions.

Expand for $\epsilon \rightarrow 0$:

$$\begin{aligned} I_\epsilon &\sim \int d\epsilon e^{-\epsilon(2k \cdot p + 1)} \epsilon^{q^2 - 2} e^{-2q^2 \sum_n \frac{w^n}{n(1-w^n)}} \left(1 - q^2 + O\left(\frac{w\epsilon^2}{(1-w)}\right) \right) \\ &\sim (2k \cdot p)^{-q^2 + 2} \Gamma(q^2) \left(1 + O\left(\frac{1}{(k \cdot p)^2}\right) \right). \end{aligned} \quad (48)$$

Being $k \cdot p = -E\sqrt{N}$ (E probe energy, \sqrt{N} tree-level mass for the massive state), our results appear to be correct in the limit of heavy massive string states and probes at high energy.

The bosonic string.

The bosonic string case is similar to the superstring one: it can be obtained eliminating from the formulas above the term $\chi(\epsilon, w)$ and substituting 1 to $g(w)$, which leads to the result:

$$I_\epsilon \sim (2k \cdot p)^{-q^2 + 2} \Gamma(q^2 - 1) \left(1 + O\left(\frac{1}{(k \cdot p)^2}\right) \right). \quad (49)$$

showing again the validity of our expansion for heavy target states and probes at high energy.

4.3 Number of string states of a given mass and size.

Ultimately, we are interested in (the logarithm of) the number of states with a given mass and size. This can be obtained from the results in section 4.2. Indeed, looking at formulas (16, 27, 28, 36) and remembering that we are considering elastic scattering, we can write:

$$F_N(\vec{q}^2) = \frac{1}{G_c(N)} \text{tr}[e^{i\vec{q} \cdot \hat{X}_{\text{closed}}^O(1)} \delta(N - \hat{N})] \quad (50)$$

where $\hat{X}_{\text{closed}}^O(z)$ is the projection of the oscillator part of the string coordinates operator, orthogonally to the momentum of the string¹⁵.

Therefore

$$\mu_N(\vec{x}) = \int \frac{dq}{(2\pi)^d} e^{-i\vec{q} \cdot \vec{x}} F_N(q^2) = \frac{1}{G_c(N)} \text{tr}[\delta(\vec{x} - \hat{X}_{\text{closed}}^O(1)) \delta(N - \hat{N})] \quad (51)$$

For fixed \vec{x} , we recognize in $\mu_N(\vec{x})$ the trace of the (incorrectly normalized) density matrix for an ensemble with fixed \vec{X}^O, \hat{N} . We note that, in terms of

¹⁵In the limit $\vec{q}^2 \rightarrow 0$, the state $|\phi\rangle$ is in his rest frame.

the string oscillators $\vec{\alpha}_m, \vec{\tilde{\alpha}}_m$:

$$\hat{X}_{\text{closed}}^O(1) = \sqrt{2} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{\vec{\alpha}_m}{m} + \frac{\vec{\tilde{\alpha}}_m}{m} \quad (52)$$

and, thanks to the normal ordering in the string amplitude, \vec{q}^2 is multiplied by

$$2 \sum_{m=1}^{\infty} \frac{\vec{\alpha}_{-m} \cdot \vec{\alpha}_m}{m} + \frac{\vec{\tilde{\alpha}}_{-m} \cdot \vec{\tilde{\alpha}}_m}{m} =: \hat{R}^2 :, \quad (53)$$

where \hat{R}^2 is the operator, modulo the zero-point contribution¹⁶, which we had described in (10), and $::$ denotes normal ordering.

Therefore, writing \vec{x} in spherical coordinates and integrating over the angular dependence, we obtain:

- for the **superstring**

- the number of closed string states with fixed R, N

$$G_c(N, R) = \frac{2}{\Gamma(\frac{d}{2})} \left(\frac{3\sqrt{d-1}}{16\pi^2\sqrt{N}} \right)^{\frac{d}{2}} \left(\frac{R}{\sqrt{N}} \right)^{d-1} \frac{e^{\pi\sqrt{d-1}\left(2\sqrt{N} - \frac{3}{16\pi^2\sqrt{N}}R^2\right)}}{N^{\frac{3}{2}}} \quad (54)$$

- and the entropy

$$\begin{aligned} S &= \ln(G_c(\sqrt{N}, R)) \\ &\sim 2\pi\sqrt{N}\sqrt{d-1} - \frac{3\sqrt{d-1}}{16\sqrt{N}\pi}R^2 + \ln\left(\frac{R^{d-1}}{\sqrt{N}^{\frac{3}{4}d+1}}\right) \end{aligned} \quad (55)$$

- for the **bosonic string**

- the number of closed string states with fixed R, N

$$G_c(N, R) = \frac{2}{\Gamma(\frac{d}{2})} \left(\frac{\sqrt{3(d-1)}}{8\sqrt{2}\pi^2\sqrt{N}} \right)^{\frac{d}{2}} \left(\frac{R}{\sqrt{N}} \right)^{d-1} \frac{e^{\pi\sqrt{d-1}\left(\sqrt{\frac{8N}{3}} - \frac{\sqrt{3}}{8\sqrt{2}\pi^2\sqrt{N}}R^2\right)}}{N^{\frac{3}{2}}} \quad (56)$$

¹⁶The zero-point contribution gives rise to the factor N^{-q^2} in (28) which is negligible for $N \rightarrow \infty$.

– and the entropy

$$\begin{aligned} S &= \ln(G_c(N, R)) \\ &\sim 4\pi\sqrt{N}\sqrt{\frac{d-1}{6}} - \frac{\sqrt{3(d-1)}}{8\sqrt{2}\sqrt{N}\pi}R^2 + \ln\left(\frac{R^{d-1}}{\sqrt{N}^{\frac{3}{4}d+1}}\right). \end{aligned} \quad (57)$$

Two remarks are important at this point. First, looking at (50, 51), we note that we have been inserting an operator

$$\delta(\vec{x} - \hat{X}_{\text{closed}}^O(1)) = \int d\vec{q} e^{i\vec{q}\cdot\vec{x} - iq\cdot\hat{X}_{\text{closed}}^O(1)} \quad (58)$$

in a string path integral. This operator, being integrated over all momenta, is off-shell. But string theory is defined only on-shell, how is then possible that our computation is correct? We appreciate here, the importance of the factorization property of (string) amplitudes: factorizing two external legs of an amplitude, the momentum square q^2 flowing along the connecting propagator is a variable, allowing analytic continuation.

Since $[\hat{X}_{\text{closed}}^O(1), \hat{N}] \neq 0$, we could also wonder whether our computation for the number of states is incorrect, because the result in (54, 56) should be independent from the ordering of the two deltas. Naturally, $\delta(\vec{x} - \hat{X}_{\text{closed}}^O(1))\delta(N - \hat{N})$ and $\delta(N - \hat{N})\delta(\vec{x} - \hat{X}_{\text{closed}}^O(1))$ yield the same result when traced over, and, furthermore, we are working with very massive string states, for which it is also reasonable to take a semi-classical limit.

5 States carrying Neveu-Schwarz charges

The results obtained in the previous sections can be extended to ensembles of string states carrying Neveu-Schwarz charges Q_R, Q_L . We have to distinguish states according to their mass and their winding and Kaluza-Klein mode numbers (m^i, n^i) , such that:

$$Q_{R,L}^i = \left(\frac{n^i}{r^i} \pm \frac{m^i r^i}{4} \right) \quad (59)$$

$$Q_{R,L}^2 = \sum_i Q_{R,L}^{i2}, \quad (60)$$

where r^i is the radius¹⁷ of compactification in the i -th compactified direction.

¹⁷Recall that we set $\alpha' = 4$ and express everything in units of α' .

The mass-shell condition and the Virasoro constraint $L_0 - \tilde{L}_0 = 0$ read:

$$M^4 = Q_L^2 + N_L \quad (61)$$

$$= Q_R^2 + N_R \quad (62)$$

$$N_L - N_R = \sum_i n^i m^i. \quad (63)$$

where L, R indicate respectively the holomorphic and anti-holomorphic sectors.

We define our microcanonical system by fixing charge and squared mass, or, more conveniently and equivalently, by constraining the values of the operators:

$$\hat{N}_L = -\hat{p}^2 - \hat{Q}_L^2 \quad \hat{N}_R = -\hat{p}^2 - \hat{Q}_R^2. \quad (64)$$

and letting their values, N_L, N_R , be large. Therefore:

- for the **superstring**

- the number of closed string states with fixed size, mass, charge is

$$G_c \sim \frac{2}{\Gamma(\frac{d}{2})} \left(\frac{3\sqrt{d-1}}{8\pi^2 \mathcal{N}} \right)^{\frac{d}{2}} \left(\frac{R}{N_L^{\frac{1}{4}} N_R^{\frac{1}{4}}} \right)^{d-1} \frac{e^{\pi\sqrt{d-1}(\mathcal{N} - \frac{3}{8\pi^2 \mathcal{N}} R^2)}}{N_L^{\frac{3}{4}} N_R^{\frac{3}{4}}} \quad (65)$$

- and the entropy

$$\begin{aligned} S &= \ln(G_c) \\ &\sim \pi \mathcal{N} \sqrt{d-1} - \frac{3\sqrt{d-1}}{8\mathcal{N}\pi} R^2 + \ln \left(\frac{R^{d-1}}{N_L^{\frac{d+2}{4}} N_R^{\frac{d+2}{4}} \mathcal{N}^{\frac{d}{2}}} \right) \end{aligned} \quad (66)$$

- for the **bosonic string**

- the number of closed string states with fixed size, mass, charge is

$$G_c = \frac{2}{\Gamma(\frac{d}{2})} \left(\frac{\sqrt{3(d-1)}}{4\sqrt{2}\pi^2 \mathcal{N}} \right)^{\frac{d}{2}} \left(\frac{R}{N_L^{\frac{1}{4}} N_R^{\frac{1}{4}}} \right)^{d-1} \frac{e^{\pi\sqrt{d-1}(\sqrt{\frac{2}{3}}\mathcal{N} - \frac{\sqrt{3}}{4\sqrt{2}\pi^2 \mathcal{N}} R^2)}}{N_L^{\frac{3}{4}} N_R^{\frac{3}{4}}} \quad (67)$$

- and the entropy

$$\begin{aligned} S &= \ln(G_c) \\ &\sim 2\pi \mathcal{N} \sqrt{\frac{d-1}{6}} - \frac{\sqrt{3(d-1)}}{4\sqrt{2}\mathcal{N}\pi} R^2 + \ln \left(\frac{R^{d-1}}{N_L^{\frac{d+2}{4}} N_R^{\frac{d+2}{4}} \mathcal{N}^{\frac{d}{2}}} \right) \end{aligned} \quad (68)$$

where we have written:

$$\mathcal{N} = \sqrt{N_L} + \sqrt{N_R}. \quad (69)$$

5.1 BPS states

We study, now, BPS configurations of fundamental superstrings. They are states with:

$$M^2 = Q_L^2, \quad N_L = 0, \quad N_R = \sum_i n^i m^i. \quad (70)$$

We find:

- the number of BPS string states with fixed size, mass, charge is

$$G_c \sim \frac{2}{\Gamma(\frac{d}{2})} \left(\frac{3\sqrt{d-1}}{8\pi^2\sqrt{N_R}} \right)^{\frac{d}{2}} \left(\frac{R}{N_R^{\frac{1}{4}}} \right)^{d-1} \frac{e^{\pi\sqrt{d-1}\left(\sqrt{N_R} - \frac{3}{8\pi^2\sqrt{N_R}}R^2\right)}}{N_R^{\frac{3}{4}}}. \quad (71)$$

- and the entropy

$$\begin{aligned} S &= \ln(G_c) \\ &\sim \pi\sqrt{N_R}\sqrt{d-1} - \frac{3\sqrt{d-1}}{8\sqrt{N_R}\pi}R^2 + \ln\left(\frac{R^{d-1}}{N_R^{\frac{d+1}{2}}}\right). \end{aligned} \quad (72)$$

It is interesting to note that the average radius for this ensemble is

$$\langle r^2 \rangle = \frac{4\pi d}{3} \sqrt{\frac{N_R}{d-1}}. \quad (73)$$

6 One-loop corrections

6.1 States with no charge

The counting of states at a given mass level is affected by the self-interaction of the string, unless we are considering supersymmetric configurations, which enjoy a protection mechanism for the mass. We are interested in counting at fixed mass and size, and therefore also the supersymmetric case will receive corrections.

Studying the mass-shift of fundamental closed strings means to compute one-loop amplitudes with the insertions of two vertex operators representing the string state. Such calculations are difficult to be performed and even defined in string theory for a series of reasons:

- the form of vertex operators for massive states is complicated

- looking for statistical properties means in principle to be able to compute one-loop two-points amplitudes for all possible string states in an ensemble, but only a few vertex operators are explicitly known
- one-loop two-points amplitudes are divergent (due to the presence of an imaginary part); they need analytic continuation, but String Theory is defined only on-shell.

An optimal method for solving these problems and computing would be factorization ([14]): starting from a known four-point amplitude, we can factorize the external legs pairwise and obtain the mass-shifts for the intermediate states as the residue of the double pole for the center of mass energy. In that case we do not need the detailed knowledge of the form of vertex operators and, as we said above, the momentum square flowing in the loop is now a variable, allowing analytical continuation. Unfortunately this approach has a residual problem: in order to identify mass-shifts for the various states we need to know the form of all their couplings with the external legs of the amplitude (for particular states, namely those on the Regge trajectory, which are non-degenerate, the method works, see [14]).

We are interested in the mass renormalization for states with both mass and size fixed. The idea is that the formulas for the entropy obtained in sections 4 and 5 will receive corrections, such that string states would have a typical size¹⁸ matching the radius of the correspondent black hole at the transition point.

The average mass-shift for states constrained in both mass and (average squared) size can be written as:

$$\Delta M_{N,R} = \frac{M^{-1}}{G_c(N, R)} \text{tr}[\widehat{\Delta M}^2 \delta(N - \hat{N}) \delta(R^2 - \hat{R}^2)] \quad (75)$$

where $\widehat{\Delta M}^2$ is an operator yielding the squared mass shift once applied to a set of states¹⁹.

Once again there is an issue in defining an operator for the observable “size”; we try to cope with this by relying on our factorization procedure, as in sections 4, 5. We consider therefore the one-loop amplitude for two

¹⁸Given by the saddle points of the integral

$$G_c(M) = \int dRe^{S(M,R)}, \quad (74)$$

where $G_c(M)$ is the total number of states at fixed squared mass M^2 .

¹⁹It can be obtained opportunely normalizing the real part of the one-loop S-matrix operator.

states represented by vertex operators V_ϕ , in the appropriate pictures for the superstring case, and two probes with vertex operator V given in (18) or the analog for the bosonic string case. Our goal is to factorize the full amplitude so that it can account for a mass-shift amplitude with the insertion of a delta function constraining the operator $\hat{X}_{\text{closed}}^O$ as in (51). We argue that, for the same reasons explained in section 4.3, this will be the most efficient way to constrain the size of the ensemble states. The relevant result, in the limit of low momentum transfer $q \equiv k + k'$ and elastic scattering, can be obtained from the OPE and integration over $\xi, \bar{\xi}$, as follows²⁰:

$$\langle V_{\phi'}(p', 0) V(k', z) V(k, \xi) V_\phi(p, \nu) \rangle_{T^2} \underset{\xi \sim z}{\underset{q^2 \rightarrow 0}{\underset{\bar{q}^2}{\sim}}} \frac{1}{q^2} \langle V_{\phi'}(p', 0) e^{i\vec{q} \cdot \hat{X}_{\text{closed}}^O(z)} |z|^{4k' \cdot \hat{p}} V_\phi(p, \nu) \rangle_{T^2} \quad (76)$$

We want to discuss mass renormalization, but the amplitude involving (76) corresponds to various field theory diagrams, accounting also for vertex corrections. To obtain the one relevant for the mass-shift, we propose to consider the limit $z \rightarrow \nu$, in order to single out the string amplitude represented in figure 2. We must be careful to take a limit where the vertex operators $V_\phi(1), V(z), V(\xi)$ approach. Indeed when they do it altogether at the same rate, we are in a situation that leads to a dangerous infrared divergence (see [19, 20, 21]). Instead, we look for a limit where two of them ($V(z), V(\xi)$) first approach each other, and then, at a slower rate, $V_\phi(\nu)$.

It is not necessary for our purposes, but it simplifies the formulas, if we work in a *time gauge* ([15, 16, 17, 18]), such that:

$$V_\phi = e^{-ip^0 X^0 + i\vec{p} \cdot \vec{X}} V(X^i) \quad \text{for the bosonic string,} \quad (77)$$

$$V_\phi = e^{-ip^0 X^0 + i\vec{p} \cdot \vec{X}} V(X^i, \psi^i, S^\alpha) \quad \text{for the fermionic string.} \quad (78)$$

Here the S^α are spin-fields, i runs over the spatial dimensions and $-p^2 = N$. Then:

$$\begin{aligned} A_{T^2}^R &= g_c^4 \iiint \langle V_{\phi'}(0) e^{i\vec{q} \cdot \hat{X}_{\text{closed}}^O(z)} V_\phi(\nu) \rangle_{T^2} \\ &\underset{\substack{z=\nu-\epsilon \\ \epsilon \rightarrow 0}}{\sim} i g_s^4 \int d^2\tau d^2\epsilon d^2\nu \frac{e^{-4\pi N \frac{\text{Im}(\nu)^2}{\text{Im}(\tau)}}}{(\text{Im}(\tau))^{\frac{d+1}{2}}} \frac{T(d_c, d, \tau, \bar{\tau})}{|\eta(\tau)|^{2(D-2)}} \left| \frac{\theta_1(\nu, \tau)}{\theta_1'(0, \tau)} \right|^{4N} |\epsilon|^{-4q^2} \\ &\quad \times \mathcal{P}_\phi^X(W, \Omega, \partial_\nu \Omega, \dots, \tilde{\Omega}, \partial_{\bar{\nu}} \tilde{\Omega}, \dots, \vec{q}B, \vec{q}^2 \partial_\epsilon B, \dots \vec{q} \tilde{B}, \vec{q}^2 \partial_{\bar{\epsilon}} \tilde{B}, \dots) \\ &\quad \times \chi(\nu, \bar{\nu}, \tau, \bar{\tau}) \end{aligned} \quad (79)$$

²⁰To avoid cluttering formulas we will write $V_\eta(v)$ instead than $V_\eta(v, \bar{v}) = \mathcal{V}_\eta(v) \mathcal{V}_\eta(\bar{v})$ for the vertex operator corresponding to a state $|\eta\rangle$. Furthermore, we have avoided, in formula (76) to write the sign of integration over $\xi, \bar{\xi}$ on the left hand side.

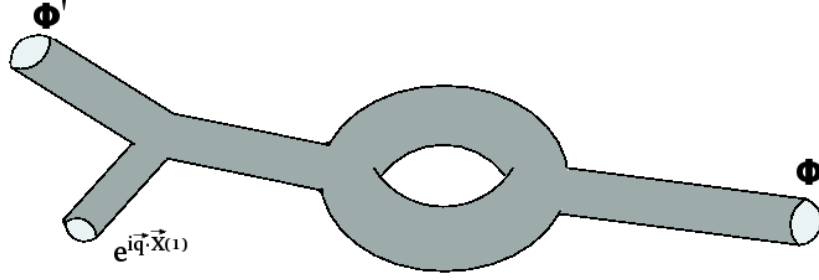


Figure 2: The diagram for the string amplitude for the one-loop mass renormalization at fixed $\hat{X}_{closed}^O(1)$.

where

$$\Omega = \partial_\nu^2 \ln(e^{-\frac{\pi \text{Im}(\nu)^2}{\text{Im}(\tau)}} \theta_1(\nu, \tau)) \quad (80)$$

$$W = \frac{2\pi}{\text{Im}(\tau)} \quad (81)$$

$$B \underset{\epsilon \rightarrow 0}{=} \partial_\epsilon \ln \left(e^{-\frac{\pi \text{Im}(\epsilon)^2}{\text{Im}(\tau)}} \theta_1(\epsilon, \tau) \right). \quad (82)$$

\mathcal{P}_ϕ is a polynomial in $W, \Omega, \partial_\nu \Omega$, higher derivatives of Ω , the anti-holomorphic $\tilde{\Omega}$ and its derivatives, B, \tilde{B} and their higher derivatives, such that:

$$\mathcal{P}_\phi^X(W, \Omega, \dots, \tilde{\Omega}, \dots, \vec{q}B, \vec{q}\tilde{B}, \dots) = \mathcal{P}_\phi(W, \Omega, \dots, \tilde{\Omega}, \dots) + \mathcal{P}'_\phi(\vec{q}B, \vec{q}\tilde{B}, \dots) \quad (83)$$

and $\chi(\nu, \bar{\nu}, \tau, \bar{\tau})$ is the fermionic part of the amplitude; the bosonic string case is recovered by substituting 1 to it. The contribution from the compactified dimensions is given by $T(d_c, d, \tau, \bar{\tau})$, which, assuming for simplicity compactification on a torus, reads:

$$T(d_c, d, \tau, \bar{\tau}) = \prod_{i=1}^{d_c-d} \frac{1}{R_i} e^{\sum_{n_i, w_i} i\tau \left(\frac{n_i}{R_i} + \frac{w_i R_i}{4} \right)^2 - i\bar{\tau} \left(\frac{n_i}{R_i} - \frac{w_i R_i}{4} \right)^2}, \quad (84)$$

where d_c is the critical (spatial) dimension, that is 25 for the bosonic theory and 9 for the superstring.

Considering the form of formula (76), together with the diagrammatic representation in figure 2, we propose to identify the amplitude (79), in the limit $z \rightarrow \nu$, with:

$$A_{T^2}^R = \langle \phi | \widehat{\Delta M}^2 \delta(R^2 - \hat{R}^2) | \phi \rangle \quad (85)$$

and from this, by tracing over states at fixed squared mass and appropriately normalizing, to arrive at (75).

The form of \mathcal{P}'_ϕ depends on the various vertex operator V_ϕ and it is very hard to say something in quantitative details about it. We content ourselves to stress the presence of the term $|\epsilon|^{-4\vec{q}^2}$ in the amplitude integrand. Indeed, let us for a moment neglect \mathcal{P}'_ϕ , we find:

$$\begin{aligned} A_{T^2}^R &\sim A_{T^2} \int_{|\epsilon| < 1} d^2\epsilon |\epsilon|^{-\vec{q}^2} \\ &\sim -\frac{A_{T^2}}{2} \frac{1}{\vec{q}^2 - 1} \end{aligned} \quad (86)$$

where A_{T^2} is the one-loop amplitude with two insertions of the vertex operator V_ϕ (giving the mass-renormalization for the state $|\phi\rangle$):

$$A_{T^2} = g_c^4 \iint \langle V_\phi(0) V_\phi(\nu) \rangle_{T^2}. \quad (87)$$

If we Fourier transform the last factor and take the real part (we are interested in the mass-shift, not in the decay), we find something of the form (J_n, Y_n are Bessel functions, a, b real functions of N):

$$-A_{T^2} \int d\vec{q} e^{i\vec{q} \cdot \vec{x}} \frac{1}{\vec{q}^2 - 1} \sim -\frac{aJ_{\frac{d-2}{2}}(|x|) + bY_{\frac{d-2}{2}}(|x|)}{|x|^{\frac{d-2}{2}}} \quad (88)$$

As a mass correction in the formulas for the entropy, this term would favour small $|x|$ (or penalize it, depending on the signs of a and b). Note also that, while the rest of the dependence on q in (79), coming from \mathcal{P}'_ϕ , depends strongly on the specific state $|\phi\rangle$ considered, this contribution will be always present, and therefore also recovered in the averaging over the states of an ensemble.

Our result is clearly incomplete, since we have neglected contributions that are not necessarily negligible in the limit $z \rightarrow \nu$. In any case it shows how terms that favour states with small average size $R = |x|$ can possibly arise. The important ingredient is the extra modulus z for the torus with punctures that we need to integrate over: this leads to the terms we are interested in. Of course, the overall sign of the amplitude is extremely important for favouring

or penalizing more compact string states. Unfortunately, due to the lack of knowledge about vertex operators for heavy massive perturbative strings, it has been impossible to estimate the full result and we leave it for future research.

6.2 Loop corrections for BPS states

For non-BPS states carrying Neveu-Schwarz charges, the consideration of the previous sections apply, once taken account of the fact that the holomorphic and the anti-holomorphic exponential part of their vertex operators are now different²¹.

The question of corrections for the BPS states is particularly subtle. We know that the counting of these states, and therefore their entropy, for fixed mass and charge do not receive corrections due to the vanishing of their two-points torus amplitude. Nevertheless, we have shown in (73) that their average size at zero coupling is larger than the string scale, and therefore of the Schwarzschild radius of the correspondent black hole at the transition point.

Which is, then, the nature of the corrections that would favour more compact states? The procedure described above fails for BPS states, because of the vanishing of the torus two-point function A_{T^2} in (86). In particular, for the states described in section 5.1, the vanishing of A_{T^2} is due to the sum over spin structure with spin-statistic signs for the holomorphic factor of the fermionic contribution. This implies that we cannot naively take the OPE as in (76), but we must *first* sum over the spin structure. With the insertion of two vertex operators of the kind given in (18), the amplitude indeed does not vanish. On the other side, we cannot propose any more what we get as a self-energy diagram with the insertion of a delta function, not even considering certain limits, as done in the previous section.

Let us see why this is the case, trying to get as close as possible to (76). The BPS states in section 5.1 are represented by the vertex operators:

$$V_\phi(z, \bar{z}) = \zeta \cdot \psi e^{ip_L \cdot X_L} e^{-i\varphi} \tilde{V}_\phi(\bar{z}) \quad \text{in the Neveu-Schwarz sector} \quad (89)$$

$$V_\phi(z, \bar{z}) = u_\alpha S^\alpha e^{ip_L \cdot X_L} e^{-\frac{i}{2}\varphi} \tilde{V}_\phi(\bar{z}) \quad \text{in the Ramond sector} \quad (90)$$

²¹Remember that our closed string states are extended only in the uncompactified dimensions, that is all their dependence on the compactified ones is in the exponential part of their vertex operator.

where:

$$\begin{aligned}
\tilde{V}_\phi(\bar{z}) & \quad \text{is the anti-holomorphic part of the vertex operator} \\
p_L \equiv (p, \vec{Q}_L) & \quad p \text{ is the momentum in the } d+1 \text{ extended dimensions} \\
S^\alpha & \quad \text{is the ground state spin field} \\
e^{-i\varphi}, e^{-\frac{i}{2}\varphi} & \quad \text{are the bosonized ground state operators for the superghost}^{22}
\end{aligned}$$

Summing over the spin structures and *then* trying to take a limit in order to reproduce at best (76), we obtain²³:

$$\begin{aligned}
\langle V_{\phi'}(p', 0) V(k', z) V(k, \xi) V_\phi(p, \nu) \rangle_{T^2} & \underset{\substack{q^2 \rightarrow 0 \\ \xi = z + |\eta| \\ |\eta| \rightarrow 0}}{\sim} \\
\frac{1}{\bar{q}^2} \langle e^{ip_L \cdot X_L(0)} \tilde{V}_{\phi'}(p', 0) e^{i\vec{q} \cdot \hat{X}_{L, \text{closed}}^O(z, \bar{z})} |z|^{4k' \cdot \hat{p}} e^{ip_L \cdot X_L(\nu)} \tilde{V}_\phi(p, \bar{\nu}) \rangle_{T^2}.
\end{aligned} \tag{91}$$

Looking at (91) we see that in any case we do not find a correlator such as (76), but instead, in the holomorphic factor, we do not have any more the (holomorphic part of the) vertex operator for the BPS state. As we said, it is therefore not possible to identify the amplitude $A_{T^2}^R$, obtained integrating over the correlator (91), with the formula

$$A_{T^2}^R = \langle \phi | \widehat{\Delta M}^2 \delta(R^2 - \hat{R}^2) | \phi \rangle \tag{92}$$

In any case, we can try to study this amplitude as a correction to the form factor, and so we find a term, by letting $z = \nu - \epsilon$, $\epsilon \rightarrow 0$:

$$\begin{aligned}
A_T^R & \underset{\substack{z = \nu - \epsilon \\ \epsilon \rightarrow 0}}{\sim} ig_c^4 \int d^2\tau d^2\epsilon d^2\nu \frac{e^{-4\pi N \frac{\text{Im}(\nu)^2}{\text{Im}(\tau)}} T(d_c, d, \tau, \bar{\tau}) \left| \frac{\theta_1(\nu, \tau)}{\theta_1'(0, \tau)} \right|^{4N}}{(\text{Im}(\tau))^{\frac{d+1}{2}} |\eta(\tau)|^{2(D-2)} \left| \frac{\theta_1'(0, \tau)}{\theta_1'(0, \tau)} \right|} |\epsilon|^{-4q^2} \\
& \quad \times \mathcal{P}_\phi^X(\tilde{\Omega}, \partial_{\bar{\nu}} \tilde{\Omega}, \dots, \tilde{q} \tilde{B}, \tilde{q}^2 \partial_\epsilon \tilde{B}, \dots) \\
& \quad \times \chi(\bar{\nu}, \bar{\tau})
\end{aligned} \tag{93}$$

which again favours compact sizes for the string, as we argued in the previous section.

²²In order to cancel the superghost charge anomaly, the vertex operator for the second insertion in the two-point function will have holomorphic part in the (-1) and $(-\frac{3}{2})$ picture, respectively for the Neveu-Schwarz and Ramond sector.

²³Formula (91) has been obtained for the case with vertex operator (89).

7 Conclusions

This work has dealt with two principal topics: the degeneracy of perturbative closed superstring states depending on their mass, charges and average size and the dynamics of such states under self-interactions.

Our principal result is the formula for the entropy of string states at fixed mass, charge and size for the free string case ($g_c = 0$). Its derivation has required the proper definitions of suitable operators and density matrices for the ensemble under investigation. The computations are well-defined within Superstring Theory, since we have obtained our results starting from well-defined string amplitudes. The key-point has been the property of factorization of the amplitudes. Indeed, string theory is defined only when the external legs of an amplitude are on-shell; however using factorization, we can operate on the momenta flowing in the internal lines, which allows analytical continuation.

The entropy formula for free string states generically suffers from corrections, due to the self-interaction of the string at non-zero coupling. We have investigated the one-loop corrections for states constrained in size as well as in mass. Unfortunately the complexity of the computation has hindered to obtain a full result, but we have shown how states with compact size can be possibly favoured by the self-interaction of the string. Our analysis has again relied on the factorization properties of well-defined string amplitudes.

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